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LETTER TO THE EDITOR

Mass renormalization in the sine-Gordon model

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Abstract. With a general Gaussian wave functional, we investigate the mass renormalization in the sine-Gordon model. At the phase transition point, the sine-Gordon system tends to a system of massless free bosons which possesses conformal symmetry.

The (1+1)-dimensional sine-Gordon model

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\phi)^2+\frac{\alpha}{\beta^2}(\cos\beta\phi-1) \tag{1}$$

has long been studied. The model is equivalent to the massive Thirring model [1], to the two-dimensional Coulomb gas [2], to the continuum limit of the lattice x-y-z spin-$\frac{1}{2}$ model [3], and to the massive O(2) nonlinear σ model [4]. Coleman discovered that the energy of the vacuum state is unbounded from below when $\beta^2>8\pi$. One may ask if there is more information to be obtained from this phase transition condition. We try to answer this question in this letter.

The sine-Gordon Hamiltonian takes the form

$$H=\int\mathscr{H}(x)\,\mathrm{d}x=\int\left\{\frac{1}{2}\pi^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2-\frac{\alpha}{\beta^2}[\cos\beta\phi-1]\right\}\mathrm{d}x. \tag{2}$$

Here we appeal to the variational approach with a general Gaussian functional

$$\begin{aligned}\Psi(\phi;\Phi,P,f)&\\ &=N_f\exp\left\{\mathrm{i}\int P(x)\phi(x)\,\mathrm{d}x-\frac{1}{2}\int\int[\phi(x)-\Phi(x)]f(x,y)\right.\\ &\quad\left.\times[\phi(y)-\Phi(y)]\,\mathrm{d}x\,\mathrm{d}y\right\}\end{aligned} \tag{3}$$

where N_f is the normalization factor, $\Phi(x)$, $P(x)$ and $f(x,y)$ are variational parameters†. The expectation value of the Hamiltonian of (2) with respect to the wave functional of (3) is given in [5] as

$$\begin{aligned}E(\Phi,P,f)=&\int\left\{\frac{1}{2}P^2+\frac{1}{2}\left(\frac{\partial\Phi}{\partial x}\right)^2-\frac{\alpha}{\beta^2}[Z\cos\beta\Phi-1]+\frac{1}{4}f(x,x)\right\}\mathrm{d}x\\ &-\frac{1}{4}\int\int\delta(x-y)\frac{\partial^2}{\partial x\partial y}f^{-1}(x,y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{4}$$

† In view of the invariance of H in (2) under the transformation of $\phi\to\phi+2n\pi/\beta$, one may construct a periodic Gaussian functional to minimize the energy. This problem requires a separate investigation elsewhere. We are grateful for the referee's remark on this point.

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where

$$Z = \exp\left\{-\frac{\beta^2}{4} f^{-1}(x, x)\right\} \tag{5}$$

and $f^{-1}(x, y)$ denotes the inverse of $f(x, y)$, i.e.

$$\int f(x, x') f^{-1}(x', y)\, \mathrm{d}x' = \delta(x - y). \tag{6}$$

The minimum-energy configuration is clearly achieved with $P(x) = 0$. As we are interested in the vicinity of some value of Φ, we set $\partial\Phi/\partial x = 0$.

For simplicity in notation we choose a function $f(x, y)$ of the form (a general $f(x, y)$ yields the same results)

$$f(x, y) = \frac{1}{2\pi} \int \mathrm{d}k \sqrt{k^2 + m^2} \cos k(x - y) \tag{7}$$

with inverse

$$f^{-1}(x, y) = \frac{1}{2\pi} \int \mathrm{d}k \frac{\cos k(x - y)}{\sqrt{k^2 + m^2}} \tag{8}$$

where m^2 is a variational parameter.

Minimizing the energy with respect to m^2 gives m^2 as a function of Φ according to following relations:

$$m^2 = \alpha Z(m^2) \cos \beta\Phi \tag{9}$$

$$Z(m^2) = \exp\left\{-\frac{\beta^2}{4} I_1(m^2)\right\} \tag{10}$$

with the notations

$$I_0(m^2) = f(x, x) = \frac{1}{2\pi} \int \mathrm{d}k \sqrt{k^2 + m^2} \tag{11}$$

$$I_1(m^2) = f^{-1}(x, x) = \frac{1}{2\pi} \int \frac{\mathrm{d}k}{\sqrt{k^2 + m^2}}. \tag{12}$$

Substituting (9)–(12) into (4) leads to the energy density ε as a function of Φ

$$\varepsilon(\Phi) = \frac{1}{2} I_0(m^2) - \frac{m^2}{4} I_1(m^2) - \frac{m^2 - \alpha}{\beta^2} \tag{13}$$

where m^2 is a function of Φ through relations (9) and (10).

Now we investigate the behaviour of this effective potential. The condition of vanishing derivative

$$\frac{\partial\varepsilon}{\partial\Phi} = \frac{m^2}{\beta} \tan(\beta\Phi) = 0 \tag{14}$$

yields

$$\beta\Phi = N\pi \qquad N = 0, \pm 1, \pm 2, \ldots . \tag{15}$$

We specialize at the vacuum sector $N = 0$, and define a mass μ by

$$\mu^2 = m^2(\Phi = 0) \tag{16}$$

which satisfies the relation

$$\mu^2 = \alpha Z(\mu^2) = \alpha \exp\left\{-\frac{\beta^2}{4} I_1(\mu^2)\right\}. \tag{17}$$

Next, the second-order derivative at $\Phi = 0$ is

$$\left.\frac{\partial^2 \varepsilon}{\partial \Phi^2}\right|_{\Phi=0} = \mu^2. \tag{18}$$

Hence the effective potential of (13) develops a minimum in the vicinity of $\Phi = 0$ as long as $\mu^2 > 0$.

Introducing an upper cutoff Λ in the integral of (12), $Z(\mu^2)$ in (10) can be explicitly evaluated as

$$Z(\mu^2) = \exp\left\{-\frac{\beta^2}{4\pi} \ln \frac{\sqrt{\alpha^{-1}\Lambda^2} + \sqrt{\alpha^{-1}\Lambda^2 + Z(\mu^2)}}{\sqrt{Z(\mu^2)}}\right\} \tag{19}$$

or

$$Z(\mu^2) = (\sqrt{\alpha^{-1}\Lambda^2} + \sqrt{\alpha^{-1}\Lambda^2 + Z(\mu^2)})^{2\gamma/(\gamma-1)} \tag{20}$$

where $\gamma = \beta^2/8\pi$. Equation (20) can be numerically solved for a pair of parameters $(\gamma, \alpha\Lambda^{-2})$; the family of curves with $Z(\mu^2) = \text{constant}$ is depicted in the parameter plane, see figure 1.

For $\alpha\Lambda^{-2} < 4$ we find that $Z(\mu^2)$ is only defined in the region $\gamma < 1$. At the vertical line $\gamma = 1$, $Z(\mu^2)$ tends to zero from the left, resulting in Coleman's transition. For

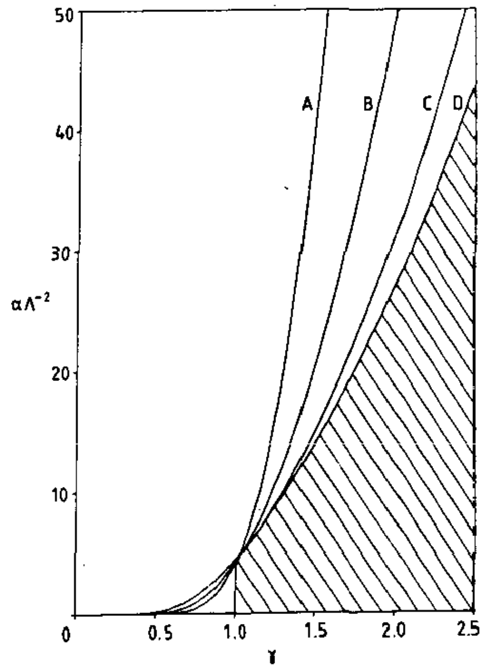

Figure 1. The phase diagram of the sine-Gordon model. The finite solutions of $Z(\mu^2)$ are in the unshaded area. The curves of A, B, C correspond to $Z(\mu^2) = 0.001$, 0.01, 0.04 respectively. The boundary D represents the envelope of the family with $Z(\mu^2) = \text{constant}$.

$\alpha\Lambda^{-2}>4$, however, equation (20) allows a finite solution for $Z(\mu^2)$ even in the region $\gamma>1$, as shown by the unshaded area in figure 1. The boundary consists of the envelope of the family $Z(\mu^2)=\text{constant}$, explicitly

$$\alpha\Lambda^{-2}=(\gamma+1)^{\gamma+1}/(\gamma-1)^{\gamma-1}. \tag{21}$$

Along the boundary the value of $Z(\mu^2)$ increases monotonously from zero at $\alpha\Lambda^{-2}=4$ to the limit value $Z(\mu^2)=e^{-2}=0.1353$. Crossing the boundary induces a kind of first-order transition. When β tends to zero, the Hamiltonian in (2) reduces to that of a free boson field with the bare mass $\mu_0=\sqrt{\alpha}$. For finite β, however, the perturbative procedure does not work. In fact the general Gaussian wave functional modifies the spectrum of the boson modes from $|k|$ to $\sqrt{k^2+\mu^2}$. For small value of $Z(\mu^2)$ we can neglect $Z(\mu^2)$ compared with $\alpha^{-1}\Lambda^2$ in the RHS of (20), resulting in a rescaled form for the mass μ,

$$\mu=\mu_0(\mu_0/2\Lambda)^{\gamma/(1-\gamma)}. \tag{22}$$

Contrasting to the semiclassical treatment, this result works for large β value, bearing a close form with the renormalized tunnelling of the spin-boson system [6].

The physical meaning of the mass μ can most convincingly be demonstrated by investigating the behaviour of the spatial correlation function

$$C(x,y)=\langle\Psi|\phi(x)\phi(y)|\Psi\rangle|_{\Phi(x)=0}$$
$$=\frac{1}{2}f^{-1}(x,y)=\frac{1}{2\pi}K_0[\mu(x-y)] \tag{23}$$

where K_0 denotes the conventional Bessel function. Then a correlation length emerges defined as $\xi=\mu^{-1}$. At long distance $|x-y|\gg\xi$ [7]

$$C(x-y)\sim[8\pi|x-y|/\xi]^{-1/2}\,e^{-|x-y|/\xi}. \tag{24}$$

When the parameters γ, $\alpha\Lambda^{-2}$ approach the vertical border $\gamma=1$, $\alpha\Lambda^{-2}<4$ from the left, the mass μ becomes vanishingly small and the space scale ξ tends to infinity, and the sine-Gordon system corresponds to the system of massless free bosons which possesses the conformal symmetry with the central charge $c=1$ [8]. Therefore Coleman's phase transition condition is also related to the conformal symmetry for such a system.

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